

# A NEW GENERALIZATION OF HERMITE'S RECIPROCITY LAW

LEANDRO CAGLIERO AND DANIEL PENAZZI

**ABSTRACT.** Given a partition  $\lambda$  of  $n$ , the *Schur functor*  $\mathbb{S}_\lambda$  associates to any complex vector space  $V$ , a subspace  $\mathbb{S}_\lambda(V)$  of  $V^{\otimes n}$ . Hermite's reciprocity law, in terms of the Schur functor, states that  $\mathbb{S}_{(p)}(\mathbb{S}_{(q)}(\mathbb{C}^2)) \simeq \mathbb{S}_{(q)}(\mathbb{S}_{(p)}(\mathbb{C}^2))$ . We extend this identity to many other identities of the type  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$ .

## 1. INTRODUCTION

Hermite's reciprocity law states that

$$\mathrm{Sym}^p(\mathrm{Sym}^q(\mathbb{C}^2)) \simeq \mathrm{Sym}^q(\mathrm{Sym}^p(\mathbb{C}^2))$$

as  $\mathrm{GL}(2, \mathbb{C})$ -modules, for any pair of non-negative integers  $p$  and  $q$ , (see e.g. [FH], Exercise 6.18). Here  $\mathrm{Sym}^n(V)$  is the homogeneous component of degree  $n$  in the symmetric algebra of  $V$ . This identity can also be stated in terms of the *Schur functor*. Recall that given any partition  $\lambda$  of  $n$ , the Schur functor  $\mathbb{S}_\lambda$  associates to any complex vector space  $V$ , a subspace (also known as the *Weyl module*)  $\mathbb{S}_\lambda(V)$  of  $V^{\otimes n}$  (see e.g. §6.1 in [FH]). We give some details in subsection §2.2). For instance, if  $\lambda = (n)$  then  $\mathbb{S}_\lambda(V) \simeq \mathrm{Sym}^n(V)$ , and if  $\lambda = (1^n)$  then  $\mathbb{S}_\lambda(V) \simeq \Lambda^n(V)$ .

Thus, in terms of Schur functors, Hermite's reciprocity law states that

$$\mathbb{S}_{(p)}(\mathbb{S}_{(q)}(\mathbb{C}^2)) \simeq \mathbb{S}_{(q)}(\mathbb{S}_{(p)}(\mathbb{C}^2)).$$

This reciprocity law has been extended to more general plethysms involving rectangle partitions by L. Manivel in [M]. More precisely a proof of Hermite's reciprocity law can be obtained from the Cayley-Sylvester formula ([Sp]); this formula was extended by M. Brion in [B] and Manivel used it to prove the following extension of Hermite's reciprocity law, valid for all positive integers  $n, k, d$ :

$$\begin{array}{ccccc} \mathbb{S}_{(n^k)}(\mathbb{S}_{(d+k-1)}(\mathbb{C}^2)) & \simeq & \mathbb{S}_{(d^n)}(\mathbb{S}_{(k+n-1)}(\mathbb{C}^2)) & \simeq & \mathbb{S}_{(k^d)}(\mathbb{S}_{(n+d-1)}(\mathbb{C}^2)) \\ \parallel & & \parallel & & \parallel \\ \mathbb{S}_{(n^d)}(\mathbb{S}_{(d+k-1)}(\mathbb{C}^2)) & \simeq & \mathbb{S}_{(d^k)}(\mathbb{S}_{(k+n-1)}(\mathbb{C}^2)) & \simeq & \mathbb{S}_{(k^n)}(\mathbb{S}_{(n+d-1)}(\mathbb{C}^2)) \end{array}$$

where the isomorphisms are now only as  $\mathrm{SL}(2, \mathbb{C})$ -modules.

It is now natural to ask for other solutions to the following plethysm equation

$$(1.1) \quad \mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$$

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considering the partitions  $\lambda$ ,  $\delta$ ,  $\mu$  and  $\epsilon$  as unknowns and the isomorphism either as  $\mathrm{SL}(2, \mathbb{C})$  or  $\mathrm{GL}(2, \mathbb{C})$ -modules.

In this paper, we obtain new solutions to the plethysm equation (1.1) involving partitions of arbitrary number of ‘steps’. Manivel’s result (involving rectangular partitions) turns out to be our one-step case. In addition, we address the question of when an  $\mathrm{SL}(2, \mathbb{C})$ -isomorphism is (or can twisted to obtain) an  $\mathrm{GL}(2, \mathbb{C})$ -isomorphism.

**Main results.** Let us denote  $\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2))$  by  $Y_{d+1}$  where  $Y$  is the Young diagram of  $\lambda$  (recall that  $\dim \mathbb{S}_{(d)}(\mathbb{C}^2) = d + 1$ ). For instance

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = \mathbb{S}_\lambda(\mathbb{S}_{(z-1)}(\mathbb{C}^2)), \quad \lambda = (3, 2^2, 1).$$

We add labels to a Young diagram to indicate the width and height of the boxes. For instance, if  $\lambda = (9^2, 5^4, 3^4)$ , its Young diagram is

$$\begin{array}{c} 3 \quad 2 \quad 4 \\ 2 \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \\ 4 \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \\ 4 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \end{array}.$$

One of the main results of the paper is the following theorem (see Theorem 3.10).

**Theorem.** Let  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  be two sequences in  $\mathbb{Z}_{\geq 0}$ , set  $|x| = \sum x_i$ ,  $|y| = \sum y_i$ , and let  $u, v, z \in \mathbb{Z}_{\geq 0}$ . Then the following  $\mathrm{SL}(2, \mathbb{C})$ -isomorphism holds:

$$\begin{array}{c} x_1 \dots x_n u \quad y_1 \dots y_n \\ \begin{array}{|c|c|c|c|c|c|} \hline x_1 & & & & & \\ \hline \vdots & & & & & \\ \hline x_n & & & & & \\ \hline v & & & & & \\ \hline y_1 & & & & & \\ \hline \vdots & & & & & \\ \hline y_n & & & & & \\ \hline \end{array} & \simeq & \begin{array}{c} x_1 \dots x_n v \quad y_1 \dots y_n \\ \begin{array}{|c|c|c|c|c|c|} \hline x_1 & & & & & \\ \hline \vdots & & & & & \\ \hline x_n & & & & & \\ \hline u & & & & & \\ \hline y_1 & & & & & \\ \hline \vdots & & & & & \\ \hline y_n & & & & & \\ \hline \end{array} \end{array}$$

$|x| + |y| + v + z \qquad \qquad \qquad |x| + |y| + u + z$

Although the diagrams in the above isomorphism have an odd number of steps, it is immediate to derive from it (taking  $u = 0$ ) the following analogous isomorphism for even number of steps:

$$\begin{array}{c} x_1 \dots x_n u \quad y_1 \dots y_{n-1} \\ \begin{array}{|c|c|c|c|c|c|} \hline x_1 & & & & & \\ \hline \vdots & & & & & \\ \hline x_n & & & & & \\ \hline v & & & & & \\ \hline y_1 & & & & & \\ \hline \vdots & & & & & \\ \hline y_{n-1} & & & & & \\ \hline \end{array} & \simeq & \begin{array}{c} x_1 \dots x_n v \quad y_1 \dots y_{n-1} \\ \begin{array}{|c|c|c|c|c|c|} \hline x_1 & & & & & \\ \hline \vdots & & & & & \\ \hline x_n & & & & & \\ \hline u & & & & & \\ \hline y_1 & & & & & \\ \hline \vdots & & & & & \\ \hline y_{n-1} & & & & & \\ \hline \end{array} \end{array}$$

$|x| + |y| + v + z \qquad \qquad \qquad |x| + |y| + u + z$

Let’s say that two pairs  $(\lambda, d)$  and  $(\mu, e)$  are equivalent if  $\mathbb{S}_\lambda(\mathbb{S}_d(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_e(\mathbb{C}^2))$ . From the above isomorphism it is possible to obtain another isomorphism by using the fact that an  $\mathrm{SL}(2, \mathbb{C})$ -module is isomorphic to its dual module. This, in general, yields an equivalence class of four different pairs  $(\lambda, d)$ . If we additionally assume in the previous theorem that  $x_i = v$

and  $y_i = z$  for all  $i = 1, \dots, n$ , then we can make use of its result twice, and obtain an equivalence class of six different pairs  $(\lambda, d)$ . In the odd case with  $n = 0$ , this equivalence class of six pairs corresponds to Manivel's Theorem.

The even and odd cases with  $n = 1$  state that

$$\begin{array}{ccc}
 \begin{array}{c} v \quad z \\ \square \quad \square \\ u \quad \square \\ u+v+z \end{array} & \simeq & \begin{array}{c} v \quad u \\ \square \quad \square \\ z \quad \square \\ v+2z \end{array} & \simeq & \begin{array}{c} u \quad z \\ \square \quad \square \\ v \quad \square \\ 2v+z \end{array} \\
 \wr & & \wr & & \wr \\
 \begin{array}{c} z \quad v \\ \square \quad \square \\ z \quad \square \\ u+v+z \end{array} & \simeq & \begin{array}{c} u \quad v \\ \square \quad \square \\ z \quad \square \\ v+2z \end{array} & \simeq & \begin{array}{c} z \quad u \\ \square \quad \square \\ v \quad \square \\ 2v+z \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{c} v \quad u \quad z \\ \square \quad \square \quad \square \\ v \quad \square \quad \square \\ z \quad \square \quad \square \\ 2v+2z \end{array} & \simeq & \begin{array}{c} v \quad v \quad z \\ \square \quad \square \quad \square \\ u \quad \square \quad \square \\ z \quad \square \quad \square \\ u+v+2z \end{array} & \simeq & \begin{array}{c} v \quad z \quad z \\ \square \quad \square \quad \square \\ u \quad \square \quad \square \\ v \quad \square \quad \square \\ u+2v+z \end{array} \\
 \wr & & \wr & & \wr \\
 \begin{array}{c} z \quad u \quad v \\ \square \quad \square \quad \square \\ z \quad \square \quad \square \\ v \quad \square \quad \square \\ 2v+2z \end{array} & \simeq & \begin{array}{c} z \quad v \quad v \\ \square \quad \square \quad \square \\ z \quad \square \quad \square \\ u \quad \square \quad \square \\ u+v+2z \end{array} & \simeq & \begin{array}{c} z \quad z \quad v \\ \square \quad \square \quad \square \\ v \quad \square \quad \square \\ u \quad \square \quad \square \\ u+2v+z \end{array}
 \end{array}$$

These, and other corollaries, are obtained in §4.

Recall that given a partition  $\lambda$  and a number  $d \geq 0$  the *hook length* of  $\lambda$  and the *d-content* of  $\lambda$  are, respectively, the following polynomials

$$\mathbf{h}_\lambda(q) = \prod [h(u)]_q, \quad \mathbf{c}_\lambda^d(q) = \prod [d+1+c(u)]_q,$$

where  $[a]_q$  is the  $q$ -analog of  $a$ ,  $h$  and  $c$  are, respectively, the hook and the content functions and both products run over the entries of the Young diagram of  $\lambda$ . It is known (see e.g. [St, Ch. 7]) that the  $\mathrm{SL}(2, \mathbb{C})$ -character of  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))$  is, up to a power of  $q$ , equal to

$$P_\lambda^d(q) = \frac{\mathbf{c}_\lambda^d(q)}{\mathbf{h}_\lambda(q)}$$

where  $d = \delta_1 - \delta_2$ .

The following theorem translates the plethysm equation (1.1) in terms of  $P$ . Although the results stated in this theorem might be known, we did not find an explicit reference to it, thus we prove it in §3 (see Theorem 3.1). If  $\lambda$  is a partition, then  $|\lambda|$  denotes the sum of its parts.

**Theorem.** Let  $\delta = (\delta_1, \delta_2)$ ,  $\epsilon = (\epsilon_1, \epsilon_2)$  and  $d = \delta_1 - \delta_2$ ,  $e = \epsilon_1 - \epsilon_2$ . Let  $\lambda, \mu$  be partitions with  $\ell(\lambda) \leq d+1$  and  $\ell(\mu) \leq e+1$ . Then

- (1)  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as  $\mathrm{SL}(2, \mathbb{C})$ -modules if and only if

$$P_\lambda^d = P_\mu^e$$

and in this case  $|\lambda|d - |\mu|e$  is even.

- (2)  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as  $\mathrm{GL}(2, \mathbb{C})$ -modules if and only if, in addition to  $P_\lambda^d = P_\mu^e$ , it also holds

$$|\delta||\lambda| = |\epsilon||\mu|.$$

## 2. TECHNICAL BACKGROUND

**2.1. Partitions.** A *partition*  $\lambda$  of  $n$  is an ordered sequence of nonnegative integers  $\lambda_1 \geq \lambda_2 \geq \dots$  with  $|\lambda| = n$ , where  $|\lambda| = \sum \lambda_i$ . The  $\lambda_i$ 's are called the parts of the partition and the length  $\ell(\lambda)$  of  $\lambda$  is the number of non zero parts. If  $k \geq \ell(\lambda)$  then  $\lambda$  will be denoted as  $\lambda = (\lambda_1, \dots, \lambda_k)$  or by indicating multiplicities with exponential notation, for instance  $(4, 4, 3, 1, 1, 1) = (4^2, 3, 1^3)$ . If  $\lambda$  and  $\mu$  are two partitions, we denote by  $\lambda + \mu$  the partition whose parts are  $(\lambda + \mu)_i = \lambda_i + \mu_i$ .

To each partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  of  $n$  we associate its *Young diagram*  $Y(\lambda)$  and its *standard tableau*  $T(\lambda)$ :  $Y(\lambda)$  is the graphical arrangement consisting of  $k$  left-justified rows of boxes, with  $\lambda_i$  boxes in the  $i$ -th row, and  $T(\lambda)$  is the assignment of the integers  $1, 2, \dots, n$  to the  $n$  boxes of  $Y(\lambda)$  obtained by writing the numbers  $1, 2, \dots, n$  starting on the first row and increasing to the right and then continuing on the second row, etc. For example, if  $\lambda = (3, 2, 2, 1)$  then

$$Y(\lambda) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} \quad T(\lambda) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & 7 & \\ \hline 8 & & \\ \hline \end{array}$$

The *transpose* of a partition is the partition  $\lambda^t$  whose Young diagram is the transpose of the Young diagram of  $\lambda$ . For example the transpose of the partition  $(3, 2, 2, 1)$  is the partition  $(4, 3, 1)$  as can be seen by the drawing above.

**2.2. Schur functor.** If  $\lambda$  is a partition of  $n$ , two subgroups of the symmetric group  $\mathfrak{S}_n$  are associated to  $T(\lambda)$ :

$$P_\lambda = \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each row of } T(\lambda)\},$$

$$Q_\lambda = \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each column of } T(\lambda)\}.$$

Following [FH] we denote by  $a_\lambda$ ,  $b_\lambda$ ,  $c_\lambda$  the following elements of the group algebra  $\mathbb{C}[\mathfrak{S}_n]$ :

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma, \quad b_\lambda = \sum_{\sigma \in Q_\lambda} \text{sgn}(\sigma)\sigma, \quad c_\lambda = a_\lambda b_\lambda.$$

The element  $c_\lambda$  is called the *Young symmetrizer* associated to  $\lambda$ . The permutation group  $\mathfrak{S}_n$  acts naturally on  $V^{\otimes n}$  by  $\sigma.(v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$ . This action is naturally extended to an action of its group algebra  $\mathbb{C}[\mathfrak{S}_n]$ . The image of  $V^{\otimes n}$  under the action of  $c_\lambda$  is denoted  $\mathbb{S}_\lambda(V)$  and the map  $V \mapsto \mathbb{S}_\lambda(V)$  is called the *Schur functor*.

For instance:

- $\mathbb{S}_{(n)}(V) \simeq \text{Sym}^n(V)$ ,
- $\mathbb{S}_{(1^n)}(V) \simeq \Lambda^n(V)$
- $\mathbb{S}_\lambda(V) = 0$  if  $\lambda$  has more than  $\dim(V)$  parts.

**2.3. Schur polynomials.** If  $\lambda$  is a partition of  $n$  and  $k \geq \ell(\lambda)$ , the *Schur polynomial* in  $k$  variables associated to  $\lambda$  is

$$s_\lambda(x_1, \dots, x_k) = \frac{\det(x_j^{\lambda_i + k - i})}{\det(x_j^{k - i})},$$

This is a symmetric polynomial in  $k$  variables of degree  $n$  for any  $k \geq \ell(\lambda)$ .

The Schur polynomial has an interesting property that will be useful later: given a partition  $\lambda$  and  $k \geq \ell(\lambda)$  we will denote by  $\lambda'$  the partition whose Young diagram is the complement of  $Y(\lambda)$  in the  $(k \times \lambda_1)$ -rectangle. (This definition depends on  $k$ , though this fact is not indicated in the notation). That is,

$$\lambda' = (\lambda_1 - \lambda_k, \dots, \lambda_1 - \lambda_2)$$

For example, for  $k = 6$  we have that

$$\text{if } Y(\lambda) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad \text{then } Y(\lambda') = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}$$

It is not difficult to prove (see Exercise 7.41 of [St]) that

$$(2.1) \quad (x_1 \dots x_k)^{\lambda_1} s_\lambda(x_1^{-1}, \dots, x_k^{-1}) = s_{\lambda'}(x_1, \dots, x_k).$$

**2.4. Polynomial representations of  $\mathrm{GL}(V)$  and  $\mathrm{SL}(V)$ .** Let  $V$  be a finite dimensional complex vector space of dimension  $k$ . A *polynomial representation* of  $\mathrm{GL}(V)$  is a finite dimensional representation of  $\mathrm{GL}(V)$  such that the matrix entries (associated to a given basis) are given by polynomial functions on  $V$ . It is well known that every polynomial representation of  $\mathrm{GL}(V)$  can be decomposed into irreducible subrepresentations. In particular,  $\mathbb{S}_\lambda(V)$  is an irreducible  $\mathrm{GL}(V)$ -subrepresentation of  $V^{\otimes n}$  for all partitions  $\lambda$  of  $n$ . The *highest weight theorem* states that  $\lambda \mapsto \mathbb{S}_\lambda(V)$  establishes a one-to-one correspondence between the set of equivalence classes of irreducible polynomial representations of  $\mathrm{GL}(V)$  and the set of partitions  $\lambda$  with  $\ell(\lambda) \leq k$ , see for instance §6 in [FH].

Moreover  $\lambda \mapsto \mathbb{S}_\lambda(V)$  also establishes a one-to-one correspondence between the set of equivalence classes of irreducible polynomial representations of  $\mathrm{SL}(V)$  and the set of partitions  $\lambda$  with  $\ell(\lambda) \leq k - 1$ . This follows from the following fact: if

$$\tilde{\lambda} = \lambda - (\lambda_k^k) = (\lambda_1 - \lambda_k, \dots, \lambda_{k-1} - \lambda_k)$$

then  $\mathbb{S}_\lambda(V) \simeq \mathbb{S}_{(\lambda_k^k)}(V) \otimes \mathbb{S}_{\tilde{\lambda}}(V)$  as  $\mathrm{GL}(V)$ -modules. But since  $\mathbb{S}_{(r^k)}(V)$  is the 1-dimensional  $\mathrm{GL}(V)$ -module corresponding to  $\det^r$ , then we obtain that  $\mathbb{S}_\lambda(V) \simeq \mathbb{S}_{\tilde{\lambda}}(V)$  as  $\mathrm{SL}(V)$ -modules. Note that  $\tilde{\lambda}$  now has at most  $k - 1$  parts.

**2.5. Characters of  $\mathrm{GL}(V)$ -modules.** If  $\pi$  is a polynomial representation of  $\mathrm{GL}(V)$ , the *character* of  $\pi$  is the function  $\chi_\pi : \mathrm{GL}(V) \rightarrow \mathbb{C}$  defined by  $\chi_\pi(g) = \mathrm{tr}(\pi(g))$ . If  $\pi_1$  and  $\pi_2$  are two polynomial representations of  $\mathrm{GL}(V)$  then  $\pi_1 \simeq \pi_2$  if and only if they have the same character. Similarly,  $\pi_1 \simeq \pi_2$  as  $\mathrm{SL}(V)$ -modules if and only if  $\chi_{\pi_1}|_{\mathrm{SL}(V)} = \chi_{\pi_2}|_{\mathrm{SL}(V)}$ . If  $g \in \mathrm{GL}(V)$  has eigenvalues  $\theta_1, \dots, \theta_k$  (counted with multiplicities), then it is known that

$$(2.2) \quad \chi_{\mathbb{S}_\lambda(V)}(g) = s_\lambda(\theta_1, \dots, \theta_k)$$

for any partition  $\lambda$  with  $\ell(\lambda) \leq k$ . (See for instance [FH]).

Let  $\delta = (\delta_1, \delta_2)$  be a partition with at most two parts and let  $d = \delta_1 - \delta_2$ . We know that  $\dim \mathbb{S}_\delta(\mathbb{C}^2) = d + 1$  and, as a representation of

$\mathrm{SL}(2, \mathbb{C})$ ,  $\mathbb{S}_\delta(\mathbb{C}^2)$  corresponds to the irreducible representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  of highest weight  $d$ .

If  $g \in \mathrm{GL}(2, \mathbb{C})$  has eigenvalues  $x_1$  and  $x_2$  then it follows from (2.2) that

$$\chi_{\mathbb{S}_\delta(\mathbb{C}^2)}(g) = s_\delta(x_1, x_2) = x_1^{\delta_1} x_2^{\delta_2} + x_1^{\delta_1-1} x_2^{\delta_2+1} + \cdots + x_1^{\delta_2} x_2^{\delta_1}.$$

Hence the eigenvalues of  $g$  in  $\mathbb{S}_\delta(\mathbb{C}^2)$  are  $\{x_1^{\delta_1} x_2^{\delta_2}, \dots, x_1^{\delta_2} x_2^{\delta_1}\}$  (all with multiplicity 1) and thus, if  $\lambda$  is a partition with  $\ell(\lambda) \leq d+1$ , then the character of  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))$  is the plethysm

$$(2.3) \quad \chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = s_\lambda(x_1^{\delta_1} x_2^{\delta_2}, x_1^{\delta_1-1} x_2^{\delta_2+1}, \dots, x_1^{\delta_2} x_2^{\delta_1}).$$

In particular, if  $g \in \mathrm{SL}(2, \mathbb{C})$  with eigenvalues  $x_1$  and  $x_1^{-1}$  then

$$\chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = \chi_{\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2))}(g) = s_\lambda(x_1^d, x_1^{d-2}, \dots, x_1^{-d}).$$

This identity and (2.1) imply that if  $\lambda'$  is as in §2.3 (with  $k = d+1$ ) then  $\chi_{\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2))}$  and  $\chi_{\mathbb{S}_{\lambda'}(\mathbb{S}_{(d)}(\mathbb{C}^2))}$  coincide in  $\mathrm{SL}(2, \mathbb{C})$  and therefore we obtain:

**Theorem 2.1.**

$$\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2)) \simeq \mathbb{S}_{\lambda'}(\mathbb{S}_{(d)}(\mathbb{C}^2))$$

as  $\mathrm{SL}(2, \mathbb{C})$ -modules.

This corresponds to the fact that  $\mathbb{S}_{\lambda'}(\mathbb{S}_{(d)}(\mathbb{C}^2))$  and  $\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2))$  are dual to each other as  $\mathrm{SL}(2, \mathbb{C})$ -modules and every polynomial representation of  $\mathrm{SL}(2, \mathbb{C})$  is isomorphic to its dual.

**2.6. The Hook-content formula.** Given a natural number  $a$ , let

$$[a] = [a]_q = \frac{1 - q^a}{1 - q} = 1 + q + \cdots + q^{a-1}$$

be the  $q$ -analog of  $a$ . If  $u = (i, j)$  is a box of the Young diagram of  $\lambda$  let  $c(u) = j - i$  and let  $h(u)$  be the number of boxes directly below or directly to the right of  $u$ , including  $u$  once. For example, we indicate in the following diagrams the values of  $c$  and  $h$  respectively:

0	1	2
-1	0	
-2	-1	
-3		

6	4	1
4	2	
3	1	
1		

Given a partition  $\lambda$  and a number  $d$  we define the *hook length* of  $\lambda$  and the *d-content* of  $\lambda$  as the following polynomials:

$$\mathbf{h}_\lambda(q) = \prod_{u \in Y(\lambda)} [h(u)]_q \quad \mathbf{c}_\lambda^d(q) = \prod_{u \in Y(\lambda)} [d + 1 + c(u)]_q.$$

Let  $\delta = (\delta_1, \delta_2)$  be a partition with at most two parts and let  $d = \delta_1 - \delta_2$ . Let  $\lambda$  be a partition with  $\ell(\lambda) \leq d+1$ . Since  $s_\lambda$  is homogeneous of degree  $|\lambda|$  it follows that

$$s_\lambda(x_1^{\delta_1} x_2^{\delta_2}, x_1^{\delta_1-1} x_2^{\delta_2+1}, \dots, x_1^{\delta_2} x_2^{\delta_1}) = (x_1^{\delta_1} x_2^{\delta_2})^{|\lambda|} s_\lambda(1, q, q^2, \dots, q^d),$$

where  $q = x_1^{-1}x_2$ . If  $b(\lambda) = \sum(i-1)\lambda_i$  and

$$P_\lambda^d(q) = \frac{\mathbf{c}_\lambda^d(q)}{\mathbf{h}_\lambda(q)}$$

then Theorem 7.21.2 in [St] states that

$$(2.4) \quad s_\lambda(1, q, \dots, q^d) = q^{b(\lambda)} P_\lambda^d(q).$$

This identity is known as the *Hook-content formula*, see the notes in Ch. 7 of [St] for more information about it.

It follows from (2.3) that if  $x_1$  and  $x_2$  are the eigenvalues of  $g \in \mathrm{GL}(2, \mathbb{C})$ , then

$$(2.5) \quad \chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = (x_1^{\delta_1} x_2^{\delta_2})^{|\lambda|} q^{b(\lambda)} P_\lambda^d(q).$$

### 3. MAIN RESULTS

**3.1. Equation (1.1) and the Hook-content formula.** The following theorem expresses the isomorphism condition of (1.1) in terms of the function  $P$ .

**Theorem 3.1.** *Let  $\delta = (\delta_1, \delta_2)$ ,  $\epsilon = (\epsilon_1, \epsilon_2)$  and  $d = \delta_1 - \delta_2$ ,  $e = \epsilon_1 - \epsilon_2$ . Let  $\lambda, \mu$  be partitions with  $\ell(\lambda) \leq d + 1$  and  $\ell(\mu) \leq e + 1$ . Then*

(1)  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as  $SL(2, \mathbb{C})$ -modules if and only if

$$(3.1) \quad P_\lambda^d = P_\mu^e$$

and in this case  $|\lambda|d - |\mu|e$  is even.

(2)  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as  $GL(2, \mathbb{C})$ -modules if and only if in addition to (3.1) it also holds

$$(3.2) \quad |\delta||\lambda| = |\epsilon||\mu|.$$

*Proof.* On the one hand, it follows from (2.5) that  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as representations of  $GL(2, \mathbb{C})$  if and only if

$$(3.3) \quad (x_1^{\delta_1} x_2^{\delta_2})^{|\lambda|} q^{b(\lambda)} P_\lambda^d(q) = (x_1^{\epsilon_1} x_2^{\epsilon_2})^{|\mu|} q^{b(\mu)} P_\mu^e(q)$$

and since the identity  $x_1 x_2 = 1$  holds in  $SL(2, \mathbb{C})$ , it follows that  $q = x_1^{-1} x_2 = x_2^2$  and hence  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as  $SL(2, \mathbb{C})$ -modules if and only if

$$(3.4) \quad x_2^{-d|\lambda|+2b(\lambda)} P_\lambda^d(x_2^2) = x_2^{-e|\mu|+2b(\mu)} P_\mu^e(x_2^2)$$

as a function of  $x_2$ .

On the other hand, since  $s_\lambda$  is symmetric, it follows from (2.3) and (2.5) that

$$x_1^{\delta_1|\lambda|-b(\lambda)} x_2^{\delta_2|\lambda|+b(\lambda)} P_\lambda^d(q) = x_2^{\delta_1|\lambda|-b(\lambda)} x_1^{\delta_2|\lambda|+b(\lambda)} P_\lambda^d(q^{-1})$$

and thus

$$\begin{aligned} \frac{P_\lambda^d(q)}{P_\lambda^d(q^{-1})} &= x_2^{(\delta_1-\delta_2)|\lambda|-2b(\lambda)} x_1^{(\delta_2-\delta_1)|\lambda|+2b(\lambda)} \\ &= q^{d|\lambda|-2b(\lambda)}. \end{aligned}$$

A similar identity holds for  $\mu$  and  $\epsilon$  instead of  $\lambda$  and  $\delta$ .

We now assume condition (3.1). This and the above identities imply that

$$(3.5) \quad d|\lambda| - 2b(\lambda) = e|\mu| - 2b(\mu)$$

and therefore (3.4) holds and thus  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as representations of  $\mathrm{SL}(2, \mathbb{C})$ . It also follows from (3.5) that  $|\lambda|d - |\mu|e$  is even.

If we additionally assume that condition (3.2) holds, then adding and subtracting (3.5) and (3.2) we obtain

$$\begin{aligned} \delta_1|\lambda| - b(\lambda) &= \epsilon_1|\tau| - b(\tau) \\ \delta_2|\lambda| + b(\lambda) &= \epsilon_2|\tau| + b(\tau), \end{aligned}$$

and taking into account that  $q = x_1^{-1}x_2$ , (3.3) follows and thus  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as representations of  $\mathrm{GL}(2, \mathbb{C})$ .

For the converse statements, we first observe that  $q = 0$  is neither a root nor a pole of the rational function  $P_\lambda^d(q) = \frac{c_\lambda^d(q)}{h_\lambda(q)}$ . Therefore, if  $\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_\epsilon(\mathbb{C}^2))$  as representations of  $\mathrm{SL}(2, \mathbb{C})$  then it follows from (3.4) that  $P_\lambda^d = P_\mu^e$ . If the isomorphism also holds as representations of  $\mathrm{GL}(2, \mathbb{C})$ , then we obtain (3.2) by specializing (3.3) at  $x_1 = x_2$ .  $\square$

**3.2.  $\mathrm{GL}(2, \mathbb{C})$ -isomorphisms from  $\mathrm{SL}(2, \mathbb{C})$ -isomorphisms.** Let  $\delta = (\delta_1, \delta_2)$ ,  $d = \delta_1 - \delta_2$ , and let  $\lambda$  be partition with  $\ell(\lambda) \leq d + 1$ . Since  $d + 1 = \dim(\mathbb{S}_{(d)}(\mathbb{C}^2))$  it follows from the discussion in §2.4, that if

$$\tilde{\lambda} = (\lambda_1 - \lambda_{d+1}, \dots, \lambda_d - \lambda_{d+1})$$

then  $\mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2)) \simeq \mathbb{S}_{\tilde{\lambda}}(\mathbb{S}_{(d)}(\mathbb{C}^2))$  as  $\mathrm{SL}(2, \mathbb{C})$ -modules. Thus, in order to study the plethysm equation (1.1) as  $\mathrm{SL}(V)$ -modules it is enough to consider the problem of finding  $d, e, \lambda$  and  $\mu$  with  $\ell(\lambda) \leq d, \ell(\mu) \leq e$ , such that

$$(3.6) \quad \mathbb{S}_\lambda(\mathbb{S}_{(d)}(\mathbb{C}^2)) \simeq \mathbb{S}_\mu(\mathbb{S}_{(e)}(\mathbb{C}^2))$$

as representations of  $\mathrm{SL}(2, \mathbb{C})$ .

On the other hand, if (3.6) holds, part (2) of Theorem 3.1 says that the isomorphism also holds as  $\mathrm{GL}(2, \mathbb{C})$ -modules if and only if  $|\lambda|d = |\mu|e$ .

If this is not the case, a natural question to ask is whether there exist  $l, m, x, y \in \mathbb{Z}_{\geq 0}$  such that

$$\mathbb{S}_{\lambda+(ld+1)}(\mathbb{S}_{(d+x)}(\mathbb{C}^2)) \simeq \mathbb{S}_{\mu+(me+1)}(\mathbb{S}_{(e+y)}(\mathbb{C}^2))$$

as representations of  $\mathrm{GL}(2, \mathbb{C})$ .

According to part (2) of Theorem 3.1 the answer is positive if and only if

$$(|\lambda| + l(d+1))(d+2x) = (|\mu| + m(e+1))(e+2y)$$

wich is equivalent to

$$(3.7) \quad (|\mu| + m(e+1))y - (|\lambda| + l(d+1))x = \frac{|\lambda|d - |\mu|e}{2} + l \binom{d+1}{2} - m \binom{e+1}{2}.$$

From part (1) of Theorem 3.1 we know that the right hand side of (3.7) is an integer number. In addition, there exist  $l, m, x, y \in \mathbb{Z}_{\geq 0}$  satisfying (3.7) if and only if there exist  $l, m \in \mathbb{Z}_{\geq 0}$  such that

$$(3.8) \quad \gcd\{(|\mu| + m(e+1)), (|\lambda| + l(d+1))\} \mid \frac{|\lambda|d - |\mu|e}{2} + l \binom{d+1}{2} - m \binom{e+1}{2}.$$



Such  $l$  and  $m$  do not always exist but in many cases they do. Concretely

**Theorem 3.2.** *If  $n$  is an integer number, let  $\nu_2(n)$  be the exponent of the highest power of the prime 2 that divides  $n$ .*

*Then there exist  $l$  and  $m$  such that (3.8) holds unless  $\nu_2(|\mu|) \neq \nu_2(|\lambda|)$  and  $0 < \min\{\nu_2(|\mu|), \nu_2(|\lambda|)\} < \min\{\nu_2(e+1), \nu_2(d+1)\}$ .*

Since this is a side issue with respect to the main thrust of this paper, and the proof, while not difficult, is slightly complicated, we will prove the above theorem in another article.

### 3.3. Equation (1.1) as $\mathbf{SL}(2, \mathbb{C})$ -modules.

**Notation 3.3.** In order to write the proofs easier, if we have a rectangular array of  $q$  numbers:

$$\overbrace{\begin{matrix} & & j \\ i \left\{ \begin{array}{cccc} [x+i+j-2] & [x+i+j-3] & \dots & [x+i-1] \\ [x+i+j-3] & [x+i+j-3] & \dots & [x+i-2] \\ & \vdots & & \vdots \\ [x+j-1] & [x+j-2] & \dots & [x] \end{array} \right. \end{matrix}}^j$$

in which all the columns and rows decrease by 1, we will denote the product of all the elements  $[*]$  in that rectangle by  $\rho_{i,j}(x)$ . Clearly  $\rho_{i,j}(x) = \rho_{j,i}(x)$  and if  $k > j$  then  $\rho_{i,k}(x) = \rho_{i,k-j}(x+j)\rho_{i,j}(x)$ .

**Lemma 3.4.** *If  $\lambda^t$  is the transpose of  $\lambda$  (see §2.1), then:*

$$\mathbf{h}_\lambda = \mathbf{h}_{\lambda^t}$$

*Proof.* Let  $x_1, \dots, x_t, y_1, \dots, y_t$  be such that the Young diagram of  $\lambda$  is:

$$Y(\lambda) = \begin{array}{c} \begin{array}{ccccc} & x_1 & x_2 & & x_t \\ y_1 & \boxed{\phantom{0}} & \boxed{\phantom{0}} & \cdots & \boxed{\phantom{0}} \\ y_2 & \boxed{\phantom{0}} & \boxed{\phantom{0}} & \cdots & \boxed{\phantom{0}} \\ & \vdots & \vdots & & \vdots \\ y_t & \boxed{\phantom{0}} & \boxed{\phantom{0}} & & \end{array} \end{array}$$

Then  $\mathbf{h}_\lambda$  is the product of all the  $\rho_{y_i, x_j}(1 + y_{i+1} + \dots + y_t + x_{j+1} + \dots + x_t)$ . Since that product is obviously symmetric on the  $x$ 's and  $y$ 's, then we obtain the result.  $\square$

**Notation 3.5.** Let  $h_1, \dots, h_t, v_1, \dots, v_{t+1}$  be positive integers. The notation  $\langle h_1, \dots, h_t | v_1, \dots, v_t, v_{t+1} \rangle$  will mean the  $\mathbf{SL}(2, \mathbb{C})$ -module  $\mathbb{S}_\lambda(\mathbb{S}_w(\mathbb{C}^2))$  where  $w = v_1 + \dots + v_{t+1} - 1$  and

$$\lambda = ((h_1 + \dots + h_{t-1} + h_t)^{v_1}, (h_1 + \dots + h_{t-1})^{v_2}, \dots, (h_1 + h_2)^{v_{t-1}}, h_1^{v_t}),$$

In order to simplify this notation, given a sequence  $x_1, x_2, \dots, x_t$ , we will denote by  $\vec{x}$  the sequence  $x_1, x_2, \dots, x_t$  and by  $\overleftarrow{x}$  the sequence  $x_t, x_{t-1}, \dots, x_1$ . That is:

$$\langle \vec{h} \parallel \vec{v} \rangle =$$

If In this notation, the  $SL(2, \mathbb{C})$ -modules isomorphism given in Theorem 2.1 becomes

**Theorem 3.6.**

$$\langle \vec{h} \parallel \vec{v} \rangle \simeq \langle \overleftarrow{h} \parallel \overleftarrow{v} \rangle$$

In pictures:

$$\begin{array}{c} \begin{array}{c} h_1 \quad h_2 \quad \dots \quad h_r \\ v_1 \\ v_2 \\ \vdots \\ v_r \\ v_1 + \dots + v_r + v_{r+1} \end{array} \quad \simeq \quad \begin{array}{c} h_r \quad h_{r-1} \quad \dots \quad h_1 \\ v_{r+1} \\ v_r \\ \vdots \\ v_2 \\ v_1 + \dots + v_r + v_{r+1} \end{array} \end{array}$$

**Theorem 3.7.** Let  $s \geq 0$  and  $t = s$  or  $t = s+1$ . Let  $x_1, \dots, x_s$  and  $y_1, \dots, y_t$  be two sequences of positive integers,  $u, v, z$  three positive integers. Let  $|\vec{x}|$  denote  $\sum_i x_i$ .

a) The following  $SL(2, \mathbb{C})$ -isomorphisms hold:

$$\begin{array}{ccc} \langle \vec{x}, u, \vec{y} \parallel z, \vec{x}, v, \vec{y} \rangle & \simeq & \langle \vec{x}, v, \vec{y} \parallel z, \vec{x}, u, \vec{y} \rangle \\ \Downarrow & & \Downarrow \\ \langle \overleftarrow{y}, u, \overleftarrow{x} \parallel \overleftarrow{y}, v, \overleftarrow{x}, z \rangle & \simeq & \langle \overleftarrow{y}, v, \overleftarrow{x} \parallel \overleftarrow{y}, u, \overleftarrow{x}, z \rangle \end{array}$$

b) Let  $S = |\vec{x}|^2 + 2 \sum_{i,j:i+j=t} x_i y_j$ . If  $z(z-1) = S + |\vec{x}|(u+v)$  in the case  $t = s$ , or  $z(z-1) = S + |\vec{x}|(u+v) + uv$  in the case  $t = s+1$  then the first row is a  $GL(2, \mathbb{C})$ -isomorphism.

c) Except in the trivial case  $u = v$ , the second row and both columns are never  $GL(2, \mathbb{C})$  isomorphisms.

*Proof.* a) The horizontal isomorphisms reveal a symmetry between  $u$  and  $v$ . Since the vertical isomorphisms follow from Theorem 3.6, we only need to prove one of the horizontal ones. We will prove the second one, i.e., we will show that  $\langle \overleftarrow{y}, u, \overleftarrow{x} \parallel \overleftarrow{y}, v, \overleftarrow{x}, z \rangle$  is symmetric on  $u$  and  $v$ .

Let us call  $\lambda_{u,v}$  the subyacent partition in  $\langle \overleftarrow{y}, u, \overleftarrow{x} \parallel \overleftarrow{y}, v, \overleftarrow{x}, z \rangle$ . Since  $\lambda_{v,u} = \lambda_{u,v}^t$ , then by Lemma 3.4 we have  $\mathbf{h}_{\lambda_{u,v}} = \mathbf{h}_{\lambda_{v,u}}$ .

Now let us see  $\mathbf{c}$ .

We need to compute  $\mathbf{c}_{\lambda_{u,v}}^w$ , where  $w = |\overleftarrow{x}| + v + |\overleftarrow{y}| + z - 1$ . Note that  $w$  depends on  $v$  but not  $u$ .

In this case the product  $\rho_{i,j}(k)$  arises from an array of the form:

$$i \left\{ \begin{array}{cccc} \overbrace{[k+i-1] \quad [k+i] \quad \dots \quad [k+i+j-2]}^j & & & \\ \vdots & & \dots & \vdots \\ [k+1] & [k+2] & \dots & [k+j] \\ [k] & [k+1] & \dots & [k+j-1] \end{array} \right.$$

Let's consider first the case  $t = s$ . The partition is then:

$$Y(\lambda_{u,v}) : \begin{array}{c} y_s \dots y_1 \quad u \quad x_s \dots x_1 \\ \begin{array}{|c|c|c|c|c|c|c|} \hline y_s & & & & & & \\ \hline \vdots & & & & & & \\ \hline y_1 & & & & & & \\ \hline v & & & & & & \\ \hline x_s & & & & & & \\ \hline \vdots & & & & & & \\ \hline x_1 & & & & & & \\ \hline \end{array} \end{array}$$

We see from  $Y(\lambda_{u,v})$  that  $\mathbf{c}_{\lambda_{u,v}}^w$  is the product of

- (1)  $\rho_{|\overleftarrow{y}|+v, |\overleftarrow{y}|+u}(w+2-|\overleftarrow{y}|-v)$ . Note that  $w+2-|\overleftarrow{y}|-v = |\overleftarrow{x}|+z+1$ , this item is  $\rho_{|\overleftarrow{y}|+v, |\overleftarrow{y}|+u}(|\overleftarrow{x}|+z+1)$ , thus symmetric in  $u, v$ .
- (2)  $\rho$ 's from the part of the table below the horizontal  $v$  line, which are independent of  $u, v$ .
- (3)  $\rho$ 's from the part of the table to the right of vertical  $u$  column. These are of the form  $\rho_{y_i, x_j}(\ast)$  and  $\ast$  is of the form  $w+2+|\overleftarrow{y}|+u$  + some  $x$ 's - some  $y$ 's, i.e.  $w+u$  + other stuff. Note that since  $w$  depends on  $v$  and not  $u$ , then  $w+u$  is symmetric on  $u, v$ .

Note that in the case  $s = t = 0$ , the proof reduces to just the case (1).

Now consider the case  $t = s + 1$ . Now the partition is:

$$Y(\lambda_{u,v}) : \begin{array}{c} y_t \dots y_1 \quad u \quad x_s \dots x_1 \\ \begin{array}{|c|c|c|c|c|c|c|} \hline y_t & & & & & & \\ \hline \vdots & & & & & & \\ \hline y_1 & & & & & & \\ \hline v & & & & & & \\ \hline x_s & & & & & & \\ \hline \vdots & & & & & & \\ \hline x_1 & & & & & & \\ \hline \end{array} \end{array}$$

As in the previous case, the  $\rho$ 's from the part of the table below the horizontal  $v$  line are independent of  $u, v$  and the  $\rho$ 's from the part of the table to the right of vertical  $u$  column depend on  $u+v$  and thus are symmetric on  $u, v$ . So the only problem is the central part of the table, which, unlike

b) Let  $\mu_{u,v}$  now denote the subyacent partition in  $\langle \vec{x}, u, \vec{y} \parallel z, \vec{x}, v, \vec{y} \rangle$  and set  $w_v = |\vec{x}| + v + |\vec{y}| + z - 1$ . By part a) of this theorem and part b) of Theorem 3.1, in order to prove  $\langle \vec{x}, u, \vec{y} \parallel z, \vec{x}, v, \vec{y} \rangle \simeq \langle \vec{x}, v, \vec{y} \parallel z, \vec{x}, u, \vec{y} \rangle$  as  $\text{GL}(2, \mathbb{C})$ -modules it suffices to see that  $|\mu_{u,v}|_{w_v} = |\mu_{v,u}|_{w_u}$ , i.e. it is enough to see that  $|\mu_{u,v}|_{w_v}$  is symmetric in  $u, v$ .

$$\begin{aligned}
|\mu_{u,v}|w_v &= \left( (|\vec{x}| + u + |\vec{y}|)z + S + |\vec{x}|(u+v) \right) \cdot (|\vec{x}| + v + |\vec{y}| + z - 1) \\
&= (|\vec{x}| + u + |\vec{y}|)z(|\vec{x}| + v + |\vec{y}|) + (|\vec{x}| + |\vec{y}|)z(z-1) + \\
&\quad + uz(z-1) + Sv + S(|\vec{x}| + |\vec{y}| + z - 1) + |\vec{x}|(u+v)v + \\
&\quad + |\vec{x}|(u+v)(|\vec{x}| + |\vec{y}| + z - 1)
\end{aligned}$$
$$uz(z-1) + Sv + |\vec{x}|((u+v)v = (u+v)(S + |\vec{x}|(u+v)))$$

Let us analyze now the case  $t = s + 1$ . The partition in this case is:

	$x_1$	$\dots$	$x_s$	$u$	$y_1$	$\dots$	$y_s$	$y_{s+1}$
$z$								
$x_1$								
$\vdots$								
$x_s$								
$v$								
$y_1$								
$\vdots$								
$y_s$								

Thus in this case  $|\mu_{u,v}| = z(|\vec{x}| + u + |\vec{y}|) + |\vec{x}|^2 + |\vec{x}|(u+v) + 2\sum_{i,j:i+j=s+1} x_i y_j + uv$ . Since in this case  $2\sum_{i,j:i+j=s+1} x_i y_j = 2\sum_{i,j:i+j=t} x_i y_j$  we have  $|\mu_{u,v}| = z(|\vec{x}| + u + |\vec{y}|) + S + |\vec{x}|(u+v) + uv$  and:

$$\begin{aligned}
|\mu_{u,v}|w_v &= \left( (|\vec{x}| + u + |\vec{y}|)z + S + |\vec{x}|(u+v) + uv \right) \cdot \left( |\vec{x}| + v + |\vec{y}| + z - 1 \right) \\
&= (|\vec{x}| + u + |\vec{y}|)z(|\vec{x}| + v + |\vec{y}|) + (|\vec{x}| + |\vec{y}|)z(z-1) + \\
&\quad + uz(z-1) + Sv + S(|\vec{x}| + |\vec{y}| + z - 1) + |\vec{x}|(u+v)v + \\
&\quad + |\vec{x}|(u+v)(|\vec{x}| + |\vec{y}| + z - 1) + uv(|\vec{x}| + |\vec{y}| + z - 1) + uv^2
\end{aligned}$$

The first, second, fifth, seventh and eighth terms are symmetric on  $u, v$ . The third, fourth, sixth and last term, using  $z(z-1) = S + |\vec{x}|(u+v) + uv$ , are equal to  $(u+v)(S + |\vec{x}|(u+v) + uv)$ , symmetric.

c) The previous second horizontal isomorphism never holds as a  $\text{GL}(2, \mathbb{C})$  isomorphism. (except in the trivial case  $u = v$ ).

This follows since  $\lambda_{v,u} = \lambda_{u,v}^t$ , hence  $|\lambda_{u,v}| = |\lambda_{v,u}|$  but  $w_v \neq w_u$  hence  $|\lambda_{u,v}|w_v \neq |\lambda_{v,u}|w_u$ , so by Theorem 3.1 the  $\text{GL}(2, \mathbb{C})$  isomorphism does not hold.  $\square$

**Remark 3.8.** Note that by the first part of Theorem 3.1,  $|\lambda_{u,v}|w_v - |\lambda_{v,u}|w_u$  must be even. Since that difference is  $|\lambda_{u,v}|(v-u)$  then either  $v-u$  is even or  $\lambda$  is even. This can also be verified directly.

**Remark 3.9.** Although in the statement and proof of Theorem 3.7 all the variables must be positive, let us see what happens if we set some of them equal to 0.

- If we set one of the variables  $y_i$  or  $x_i$  equal to zero, what happens is that this gives rise to another configuration with all variables positive but both  $s$  and  $t$  decrease by 1. For example, if we set  $x_1 = 0$ , this eliminates an  $x$  variable, decreasing  $s$  by 1 but “joins”  $y_{s-1}$  and  $y_s$  to form a new variable with value  $y_{s-1} + y_s$ , thus decreasing  $s$  by 1 too. Hence the total number of variables decrease by two.
- If we set the variable  $z = 0$ , then we decrease the total number of variables by 1, and we switch from the  $t = s$  case to the  $t = s+1$  case

and viceversa, but we go for example from the upper isomorphism of case  $s = t$  to the lower isomorphism for case  $t = s + 1$ .

Therefore we could state just one theorem, in the following form:

**Theorem 3.10.** *Let  $s \geq 0$ . Let  $x_1, \dots, x_s$  and  $y_1, \dots, y_s$  be two sequences of nonnegative integers,  $u, v, z$  three nonnegative integers. Then the following  $SL(2, \mathbb{C})$ -isomorphisms hold:*

$$\begin{array}{ccc} \langle \vec{x}, u, \vec{y} \parallel z, \vec{x}, v, \vec{y} \rangle & \simeq & \langle \vec{x}, v, \vec{y} \parallel z, \vec{x}, u, \vec{y} \rangle \\ \Downarrow & & \Downarrow \\ \langle \overleftarrow{y}, u, \overleftarrow{x} \parallel \overleftarrow{y}, v, \overleftarrow{x}, z \rangle & \simeq & \langle \overleftarrow{y}, v, \overleftarrow{x} \parallel \overleftarrow{y}, u, \overleftarrow{x}, z \rangle \end{array}$$

#### 4. SOME COROLLARIES

Here we obtain corollaries of Theorem 3.7.

**Remark 4.1.** Hermite's is a corollary of our theorem, since it is the case  $s = t = 0$ , with  $z = 1$  which implies that the condition of part b) of Theorem 3.7 is satisfied, since  $z(z - 1) = 0$  while  $S = |\vec{x}| = 0$  too. Hence we obtain the full statement of Hermite's, while from Manivel's result only the  $SL(2, \mathbb{C})$  isomorphism can be deduced.

**Theorem 4.2.** *Let  $v, z, u$  be positive integers. Let  $s \geq 0$ . Then the two following families of isomorphism hold:*

$$\begin{array}{ccc} \langle z^s v^{s+1} \parallel z^{s+1} u v^s \rangle & \simeq & \langle z^s u v^s \parallel z^{s+1} v^{s+1} \rangle \simeq \langle z^{s+1} v^s \parallel z^s u v^{s+1} \rangle \\ \text{(I)} \quad \Downarrow & & \Downarrow \quad \Downarrow \\ \langle v^s z^{s+1} \parallel v^{s+1} u z^s \rangle & \simeq & \langle v^s u z^s \parallel v^{s+1} z^{s+1} \rangle \simeq \langle v^{s+1} z^s \parallel v^s u z^{s+1} \rangle \end{array}$$

and

$$\begin{array}{ccc} \langle z^s u v^{s+1} \parallel z^{s+2} v^{s+1} \rangle & \simeq & \langle z^{s+1} v^{s+1} \parallel z^{s+1} u v^{s+1} \rangle \simeq \langle z^{s+1} u v^s \parallel z^{s+1} v^{s+2} \rangle \\ \text{(II)} \quad \Downarrow & & \Downarrow \quad \Downarrow \\ \langle v^s u z^{s+1} \parallel v^{s+2} z^{s+1} \rangle & \simeq & \langle v^{s+1} z^{s+1} \parallel v^{s+1} u z^{s+1} \rangle \simeq \langle v^{s+1} u z^s \parallel v^{s+1} z^{s+2} \rangle \end{array}$$

*Proof.* From Theorem 3.7 we obtain:

$$\begin{array}{ccc} \langle z^s u v^s \parallel z^{s+1} v^{s+1} \rangle & \simeq & \langle z^s v^{s+1} \parallel z^{s+1} u v^s \rangle \\ \Downarrow & & \Downarrow \\ \langle v^s u z^s \parallel v^{s+1} z^{s+1} \rangle & \simeq & \langle v^{s+1} z^s \parallel v^s u z^{s+1} \rangle \end{array}$$

If we apply the lower isomorphism to  $\langle z^s u v^s \parallel z^{s+1} v^{s+1} \rangle$  we obtain:

$$\begin{array}{ccc} \langle z^{s+1} v^s \parallel z^s u v^{s+1} \rangle & \simeq & \langle z^s u v^s \parallel z^{s+1} v^{s+1} \rangle \simeq \langle z^s v^{s+1} \parallel z^{s+1} u v^s \rangle \\ \Downarrow & & \Downarrow \\ \langle v^s u z^s \parallel v^{s+1} z^{s+1} \rangle & \simeq & \langle v^{s+1} z^s \parallel v^s u z^{s+1} \rangle \end{array}$$

Applying theorem 3.6 to the top left, we get:

$$\begin{array}{ccc}
\langle z^{s+1}v^s || z^s uv^{s+1} \rangle & \simeq & \langle z^s uv^s || z^{s+1}v^{s+1} \rangle \simeq \langle z^s v^{s+1} || z^{s+1} uv^s \rangle \\
\Downarrow & & \Downarrow \qquad \qquad \qquad \Downarrow \\
\langle v^s z^{s+1} || v^{s+1} uz^s \rangle & \simeq & \langle v^s uz^s || v^{s+1} z^{s+1} \rangle \simeq \langle v^{s+1} z^s || v^s uz^{s+1} \rangle
\end{array}$$

Rearranging the top line we get the first result. The second result is similar: from Theorem 3.7 we get:

$$\begin{array}{ccc}
\langle z^{s+1}v^{s+1} || z^{s+1} uv^{s+1} \rangle & \simeq & \langle z^s uv^{s+1} || z^{s+2} v^{s+1} \rangle \\
\Downarrow & & \Downarrow \\
\langle v^{s+1} z^{s+1} || v^{s+1} uz^{s+1} \rangle & \simeq & \langle v^{s+1} uz^s || v^{s+1} z^{s+2} \rangle
\end{array}$$

Again, applying the lower isomorphism to  $\langle z^{s+1}v^{s+1} || z^{s+1} uv^{s+1} \rangle$ , using theorem 3.6 and rearranging the top line we get the result.  $\square$

**Remark 4.3.** Note that the isomorphism  $I$  with  $s = 0$  gives:

$$\begin{array}{ccc}
\langle v || zu \rangle & \simeq & \langle u || zv \rangle \simeq \langle z || vu \rangle \\
\Downarrow & & \Downarrow \qquad \qquad \qquad \Downarrow \\
\langle z || vu \rangle & \simeq & \langle u || vz \rangle \simeq \langle v || zu \rangle
\end{array}$$

i.e., it says that  $\langle z || vu \rangle$  is symmetric in  $z, v, u$ . This is Manivel's result. It is not possible to say more because in order to apply 3.7 b) in this case, we would need  $z(z-1) = 0$  which only happens when  $z = 1$ , and this is Hermite's.

**Remark 4.4.** The isomorphism  $II$  with  $s = 0$  gives:

$$\begin{array}{ccc}
\langle uv || z zv \rangle & \simeq & \langle zv || zuv \rangle \simeq \langle zu || z vv \rangle \\
\Downarrow & & \Downarrow \qquad \qquad \qquad \Downarrow \\
\langle uz || v v z \rangle & \simeq & \langle v z || v u z \rangle \simeq \langle v u || v z z \rangle
\end{array}$$

and the top-right isomorphism is a  $\text{GL}(2, \mathbb{C})$  isomorphism if  $z(z-1) = uv$ .

**Remark 4.5.** If  $s \geq 1$  we cannot obtain  $\text{GL}(2, \mathbb{C})$  isomorphisms from the isomorphisms I or II.

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CIEM-FAMAF, UNIVERSIDAD NACIONAL DE CÓRDOBA  
*E-mail address:* `cagliero@famaf.unc.edu.ar`

CIEM-FAMAF, UNIVERSIDAD NACIONAL DE CÓRDOBA  
*E-mail address:* `penazzi@famaf.unc.edu.ar`